On q-Rising Factorials and the Necessity of Shared Memory Calculations

Ali Kemal Uncu (akuncu@ricam.oeaw.ac.at)

Johann Radon Institute for Computational and Applied Mathematics

Austrian Academy of Sciences

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Partitions

A *partition* π is a finite sequence of *non-decreasing* positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_{\#(\pi)})$.

For a given partition $\pi = (\lambda_1, \lambda_2, \ldots, \lambda_{\#(\pi)})$ the sum $\lambda_1 + \lambda_2 + \cdots + \lambda_{\#(\pi)}$ is the *size* of the partition π and it is denoted by $|\pi|$.

Ex:

- $\pi = (1, 1, 5)$ is a partition of $|\pi| = 7$.
- $\pi = \emptyset$ is the unique partition of 0.

Generating Functions

For a sequence $\{a_n\}_{n=0}^{\infty}$, the series

$$\sum_{n\geq 0} a_n q^n$$

is called a *generating function*.

Let $\mathcal{D}$ be the set of all partitions into non-repeating parts.

$$\sum_{\pi\in\mathcal{D}} q^{|\pi|} = 1+q+q^2+2q^3+2q^4+3q^5+4q^6+5q^7+6q^8+8q^9\ldots.$$

$$\begin{array}{c|l|l}
\emptyset & (1,2),\ (3) & (1,2,3),\ (1,5),\ (2,4),\ (6) \\
(1) & (1,3),\ (4) & (1,2,4),\ (1,6),\ (2,5),\ (3,4),\ (7) \\
(2) & (1,4),\ (2,3),\ (5) & (1,2,5),\ (1,3,4),\ (1,7),\ (2,6),\ (3,5),\ (8)
\end{array}$$

q-Pochhammer Symbol

$$(a;q)_L := \prod_{i=0}^{L-1}(1-aq^i), \text{ and } (a;q)_\infty := \lim_{L\to\infty}(a;q)_L.$$

$$\begin{aligned}(-q;q)_\infty &= (1+q^1)(1+q^2)(1+q^3)\dots\\ &= 1+q^1+q^2+(q^{1+2}+q^3)+\dots\\ &= \sum_{\pi\in\mathcal{D}} q^{|\pi|},\end{aligned}$$

where $\mathcal{D}$ is the set of all partitions into non-repeating parts. Similarly,

$$(q;q)_\infty = \sum_{\pi\in\mathcal{D}} (-1)^{\#(\pi)} q^{|\pi|}.$$

Generating Functions

$$\sum_{\pi \in \mathcal{D}} (-1)^{\#(\pi)} q^{|\pi|} = \sum_{\substack{\pi \in \mathcal{D} \\ \#(\pi) \equiv 0 \bmod 2}} q^{|\pi|} - \sum_{\substack{\pi \in \mathcal{D} \\ \#(\pi) \equiv 1 \bmod 2}} q^{|\pi|}.$$

$$\begin{array}{c|l|l} \emptyset & (1,2),\ (3) & (1,2,3),\ (1,5),\ (2,4),\ (6) \\ (1) & (1,3),\ (4) & (1,2,4),\ (1,6),\ (2,5),\ (3,4),\ (7) \\ (2) & (1,4),\ (2,3),\ (5) & (1,2,5),\ (1,3,4),\ (1,7),\ (2,6),\ (3,5),\ (8) \end{array}$$

Generating Functions

$$\sum_{\pi\in\mathcal{D}} (-1)^{\#(\pi)} q^{|\pi|} = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \dots .$$

$\emptyset$	$(1,2)$, (3)	$(1,2,3)$, $(1,5)$, $(2,4)$, (6)
(1)	$(1,3)$, (4)	$(1,2,4)$, $(1,6)$, $(2,5)$, $(3,4)$, (7)
(2)	$(1,4)$, $(2,3)$, (5)	$(1,2,5)$, $(1,3,4)$, $(1,7)$, $(2,6)$, $(3,5)$, (8)

Theorem (Euler's Pentagonal Number Theorem, 1750s)

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}.$$

Theorem (Euler's Pentagonal Number Theorem, 1750s)

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}.$$

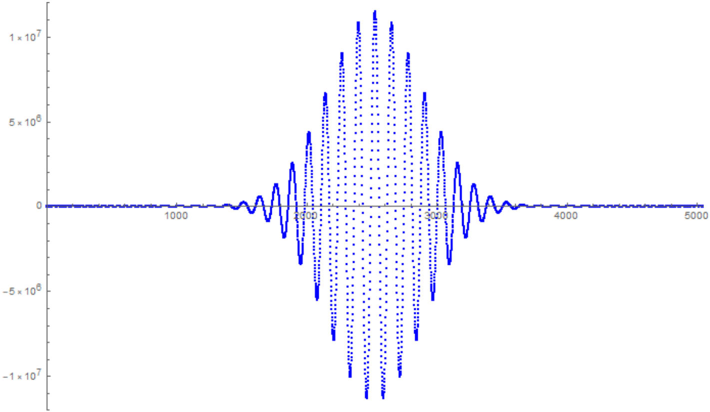
Let N be a positive integer,

$$(q;q)_N := \sum_{i=0}^{N(N+1)/2} a_{i,N}\, q^i.$$

What about the $a_{i,N}$?

$$(q;q)_4 = 1 - q - q^2 + 2q^5 - q^8 - q^9 + q^{10}.$$

Plot of $(q;q)_{100}$



Maximum absolute coefficient is $a_{2525,100} = 11,493,312.$

Classification of $(q;q)_N$ by maximum $|a_{i,N}|$

Joint with Berkovich, I developed an elementary method to find all N where $|a_{i,N}| < L$ for any choice of L.[1]

While looking through the literature, we saw the following:

[1] On some polynomials of Bloch–Pólya type, with A. Berkovich, Proc. Amer. Math. Soc. 146 (2018), no. 7, 2827–2838

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THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A160089 The maximum of the absolute value of the coefficients of Pn = (1-x)(1-x^2)(1-x^3)...(1-x^n). 4

1, 1, 1, 1, 2, 1, 2, 2, 2, 2, 3, 2, 4, 3, 3, 4, 6, 5, 6, 7, 8, 8, 10, 11, 16, 16, 19, 21, 28, 29, 34, 41, 50, 56, 68, 80, 100, 114, 135, 158, 196, 225, 269, 320, 388, 455, 544, 644, 786, 921, 1111, 1321, 1600, 1891, 2274, 2711, 3280, 3895, 4694, 5591, 6780, 8051, 9729, 11624 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,5

COMMENTS If n is even then a(n) is the absolute value of the coefficient of z^(n(n+1)/4). If n is odd, it is an open question as to which coefficient is a(n).

Where is the maximum abs. coefficient of $(q;q)_n$?

The only way to speculate is to calculate.
Recall:

$$
\begin{aligned}
(q;q)_N &:= (1-q)(1-q^2)\dots(1-q^{N-1})(1-q^N) \\
&= \sum_{i=0}^{N(N+1)/2} a_{i,N}\ q^i.
\end{aligned}
$$

Therefore,

$$a_{i,N} = a_{i,N-1} - a_{i-N,N-1}.$$

Necessary resources for the calculation

That's where a big computer like MACH2[2] becomes a necessity!

- The degree(length) of the polynomial(array) grows quadratically.
- The magnitude of the maximum coefficient grows exponentially[3].
- To parallelize efficiently, we need to at least double the amount of data used.
- Splitting the data makes no sense.

[2]Please feel free to ask me about this supercomputer after the talk.

[3]C. Sudler Jr, *Two algebraic identities and the unboundedness of a restricted partition function*. Proc. Amer. Math. Soc. 15 (1964), 16-20.

What did we calculate?

$$(q;q)_N := \sum_{i=0}^{N(N+1)/2} a_{i,N}\ q^i,$$

define

$$\mathcal{M}_N = \max_i |a_{i,N}|$$

and $L(N)$ be the smallest integer such that $a_{L(N),N} = \mathcal{M}_N$. Let

$$D_N = \frac{N(N+1)}{4} - L(N) \text{ and } E_N = D_N - D_{N-4}.$$

What did we calculate?

- $L(N)$ location of the maximum absolute coefficient,
- D_N distance of the maximum absolute coefficient from the midpoint,
- $E_N = D_N - D_{N-4}$ the first difference of the D_N's in residue classes mod 4.

Conjecture (Berkovich-U. 2019)

$E_N \in \{1, 2\}$ *for all* $N \geq 61$.

$$L(N) = \frac{N(N+1)}{4} - \left(D_m + \sum_{i=1}^{(N-m)/4} E_{m+4i} \right).$$

E_N's are almost periodic with period 19

Starting from $N = 53$, the E_N's for 1 mod 4 parts look as follows:

$$2112111211121112111$$
$$2112111211121112111$$
$$2112111211121112112$$
$$1112111211121112112$$
$$1112111211121112112$$
$$1112111211121112112$$
$$1112111211121112112$$
$$1112111211121121112$$
$$1112111211121121112$$
$$1112111211121121112$$
$$\ldots\ldots$$

E_N's are almost periodic with period 19

Starting from $N = 53$, the E_N's for 1 mod 4 parts look as follows:

2112111211121112111

2112111211121112111

2112111211121112112

1112111211121112112

1112111211121112112

1112111211121112112

1112111211121112112

1112111211121121112

1112111211121121112

1112111211121121112

......

The alphabet

$$
\begin{aligned}
a &= \overbrace{2112111211121112112}^{19}, & k &= \overbrace{1211121112111211121}^{19},\\
b &= 1112111211121112112, & l &= 1211121112111211211,\\
c &= 1112111211121121112, & m &= 1211121112112111211,\\
d &= 1112111211211121112, & n &= 1211121121112111211,\\
e &= 1112112111211121112, & o &= 1211211121112111211,\\
f &= 1121112111211121112, & p &= 2111211121112111211,\\
g &= 1121112111211121121, & q &= 2111211121112112111,\\
h &= 1121112111211211121, & r &= 2111211121121112111,\\
i &= 1121112112111211121, & s &= 2111211211121112111,\\
j &= 1121121112111211121, & t &= 2112111211121112111.
\end{aligned}
$$

The words using this alphabet

$$\underbrace{a^1b^4c^4d^4e^4f^3g^4h^5i^4j^4k^3l^4m^4n^4o^5p^3q^4r^4s^4t^3}_{\text{word 1}}\underbrace{a^1b^4c^4d^4e^4f^3g^4h^4i^5j^4k^3\ldots}_{\text{word 2, etc.}}.$$

$$\begin{array}{l}
a^1\,b^4\,c^4\,d^4\,e^4\,f^3\,g^4\,h^5\,i^4\,j^4\,k^3\,l^4\,m^4\,n^4\,o^5\,p^3\,q^4\,r^4\,s^4\,t^3\\
a^1\,b^4\,c^4\,d^4\,e^4\,f^3\,g^4\,h^4\,i^5\,j^4\,k^3\,l^4\,m^4\,n^4\,o^5\,p^3\,q^4\,r^4\,s^4\,t^3\\
a^1\,b^3\,c^5\,d^4\,e^4\,f^3\,g^4\,h^4\,i^5\,j^4\,k^3\,l^4\,m^4\,n^4\,o^4\,p^4\,q^4\,r^4\,s^4\,t^3\\
a^1\,b^3\,c^5\,d^4\,e^4\,f^3\,g^4\,h^4\,i^4\,j^5\,k^3\,l^4\,m^4\,n^4\,o^4\,p^4\,q^4\,r^4\,s^4\,t^3\\
a^1\,b^3\,c^4\,d^5\,e^4\,f^3\,g^4\,h^4\,i^4\,j^5\,k^3\,l^4\,m^4\,n^4\,o^4\,p^3\,q^5\,r^4\,s^4\,t^3\\
a^1\,b^3\,c^4\,d^5\,e^4\,f^3\,g^4\,h^4\,i^4\,j^4\,k^4\,l^4\,m^4\,n^4\,o^4\,p^3\,q^4\,r^5\,s^4\,t^3\\
a^1\,b^3\,c^4\,d^4\,e^5\,f^3\,g^4\,h^4\,i^4\,j^4\,k^3\,l^5\,m^4\,n^4\,o^4\,p^3\,q^4\,r^5\,s^4\,t^3\\
a^1\,b^3\,c^4\,d^4\,e^4\,f^4\,g^4\,h^4\,i^4\,j^4\,k^3\,l^5\,m^4\,n^4\,o^4\,p^3\,q^4\,r^4\,s^5\,t^3\\
a^1\,b^3\,c^4\,d^4\,e^4\,f^4\,g^4\,h^4\,i^4\,j^4\,k^3\,l^4\,m^5\,n^4\,o^4\,p^3\,q^4\,r^4\,s^5\,t^3\\
a^1\,b^3\,c^4\,d^4\,e^4\,f^3\,g^5\,h^4\,i^4\,j^4\,k^3\,l^4\,m^5\,n^4\,o^4\,p^3\,q^4\,r^4\,s^4\,t^4\\
a^1\,b^3\,c^4\,d^4\,e^4\,f^3\,g^5\,h^4\,i^4\,j^4\,k^3\,l^4\,m^4\,n^5\,o^4\,p^3\,q^4\,r^4\,s^4\,t^4\\
a^1\,b^3\,c^4\,d^4\,e^4\,f^3\,g^4\,h^5\,i^4\,j^4\,k^3\,l^4\,m^4\,n^5\,o^4\,p^3\,q^4\,r^4\,s^4\,t^3\\
a^1\,b^4\,c^4\,d^4\,e^4\,f^3\,g^4\,h^4\,i^5\,j^4\,k^3\,l^4\,m^4\,n^4\,o^5\,p^3\,q^4\,r^4\,s^4\,t^3\\
a^1\,b^3\,c^5\ldots
\end{array}$$

The words using this alphabet

$$\underbrace{a^1 b^4 c^4 d^4 e^4 f^3 g^4 h^5 i^4 j^4 k^3 l^4 m^4 n^4 o^5 p^3 q^4 r^4 s^4 t^3}_{\text{word } 1}\underbrace{a^1 b^4 c^4 d^4 e^4 f^3 g^4 h^4 i^5 j^4 k^3 \ldots}_{\text{word } 2, \text{ etc.}}.$$

$$\begin{array}{c}
a^1\, b^4\, c^4\, d^4\, e^4\, f^3\, g^4\, h^5\, i^4\, j^4\, k^3\, l^4\, m^4\, n^4\, o^5\, p^3\, q^4\, r^4\, s^4\, t^3 \\
\hline
11 \left\{ \begin{array}{c}
a^1\, b^4\, c^4\, d^4\, e^4\, f^3\, g^4\, h^4\, i^5\, j^4\, k^3\, l^4\, m^4\, n^4\, o^5\, p^3\, q^4\, r^4\, s^4\, t^3 \\
a^1\, b^3\, c^5\, d^4\, e^4\, f^3\, g^4\, h^4\, i^5\, j^4\, k^3\, l^4\, m^4\, n^4\, o^4\, p^4\, q^4\, r^4\, s^4\, t^3 \\
a^1\, b^3\, c^5\, d^4\, e^4\, f^3\, g^4\, h^4\, i^4\, j^5\, k^3\, l^4\, m^4\, n^4\, o^4\, p^4\, q^4\, r^4\, s^4\, t^3 \\
a^1\, b^3\, c^4\, d^5\, e^4\, f^3\, g^4\, h^4\, i^4\, j^5\, k^3\, l^4\, m^4\, n^4\, o^4\, p^3\, q^5\, r^4\, s^4\, t^3 \\
a^1\, b^3\, c^4\, d^5\, e^4\, f^3\, g^4\, h^4\, i^4\, j^4\, k^4\, l^4\, m^4\, n^4\, o^4\, p^3\, q^4\, r^5\, s^4\, t^3 \\
a^1\, b^3\, c^4\, d^4\, e^5\, f^3\, g^4\, h^4\, i^4\, j^4\, k^3\, l^5\, m^4\, n^4\, o^4\, p^3\, q^4\, r^5\, s^4\, t^3 \\
a^1\, b^3\, c^4\, d^4\, e^4\, f^4\, g^4\, h^4\, i^4\, j^4\, k^3\, l^5\, m^4\, n^4\, o^4\, p^3\, q^4\, r^4\, s^5\, t^3 \\
a^1\, b^3\, c^4\, d^4\, e^4\, f^4\, g^4\, h^4\, i^4\, j^4\, k^3\, l^4\, m^5\, n^4\, o^4\, p^3\, q^4\, r^4\, s^5\, t^3 \\
a^1\, b^3\, c^4\, d^4\, e^4\, f^3\, g^5\, h^4\, i^4\, j^4\, k^3\, l^4\, m^5\, n^4\, o^4\, p^3\, q^4\, r^4\, s^4\, t^4 \\
a^1\, b^3\, c^4\, d^4\, e^4\, f^3\, g^5\, h^4\, i^4\, j^4\, k^3\, l^4\, m^4\, n^5\, o^4\, p^3\, q^4\, r^4\, s^4\, t^4 \\
a^1\, b^3\, c^4\, d^4\, e^4\, f^3\, g^4\, h^5\, i^4\, j^4\, k^3\, l^4\, m^4\, n^5\, o^4\, p^3\, q^4\, r^4\, s^4\, t^3
\end{array} \right. \\
\hline
a^1\, b^4\, c^4\, d^4\, e^4\, f^3\, g^4\, h^4\, i^5\, j^4\, k^3\, l^4\, m^4\, n^4\, o^5\, p^3\, q^4\, r^4\, s^4\, t^3 \\
a^1\, b^3\, c^5 \ldots
\end{array}$$

Conjecture (Berkovich-U. 2019[4])

For any $k \geq 0$, $r = 0, 1, 2, \ldots 10$, *let*

$$A(k,r) = 5700r + 62624k - 76\,\delta_{r,11},$$

where $\delta_{i,j}$ *is the Kronecker delta, then,*

$$L(5909 + A(k,r)) = \frac{(5909 + A(k,r))(5910 + A(k,r))}{4} - D_{5909+A(0,r)} - 19787k,$$

where the full list of needed seed values of D_n*'s are*

$$\begin{aligned} D_{5909} &= 1867.5, & D_{11609} &= 3668.5, & D_{17309} &= 5469.5, \\ D_{23009} &= 7270.5, & D_{28709} &= 9071.5, & D_{34409} &= 9675.5, \\ D_{40109} &= 12673.5, & D_{45809} &= 14474.5, & D_{51509} &= 16275.5, \\ D_{57209} &= 18076.5, & D_{62833} &= 19853.5. \end{aligned}$$

[4] Where do the maximum absolute q-series coefficients of $(1-q)(1-q^2)(1-q^3)\ldots(1-q^{n-1})(1-q^n)$ occur?, with A. Berkovich.

Thank you for your time

and a special thanks to the MACH2 group.

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